Project #15

Properties of Continuous Time Fourier Series, Conjugation and Conjugate Symmetry

In this project, we will demonstrate conjugation and symmetry properties of continuous Fourier series (FS). When we take the complex conjugate of a periodic signal $x(t)$, corresponding FS coefficients will be time reversed, i.e. flipped about $k = 0$, and complex conjugated. We can express this property as follows.

$$x(t) \xleftrightarrow{FS} a_k \Leftrightarrow x^*(t) \xleftrightarrow{FS} a^*_{-k} \tag{15.1}$$

We can easily prove this property by taking complex conjugate of the both sides of the reconstruction equation.

$$\begin{aligned} x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} \Rightarrow x^*(t) &= \left(\sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} \right)^* \\ &= \sum_{k=-\infty}^{+\infty} a^*_k e^{-jkw_0 t} \end{aligned} \tag{15.2}$$

By making the substitution $m = -k$ a change in equation (15.2) we obtain,

$$x^*(t) = \sum_{k=-\infty}^{+\infty} a^*_k e^{-jkw_0 t} = \sum_{m=+\infty}^{-\infty} a^*_{-m} e^{jmw_0 t} = \sum_{m=-\infty}^{+\infty} a^*_{-m} e^{jmw_0 t} . \tag{15.3}$$

Equation (15.3) already expresses $x^*(t)$ in FS from, where we see that that FS coefficients are a^*_{-k}.

To construct our demonstration we used a complex periodic signal $z(t)$ that has a cosine $x(t)$ and a square wave $y(t)$ as its real and imaginary parts respectively. This signal is shown in Figure 15.1 (upper panels). From previous projects, we both know the FS representations of cosine and periodic square wave signals. These representations are given by equations (15.5) and (15.6).

$$z(t) = x(t) + jy(t)$$
$$x(t) = \cos(2\pi t) \qquad (15.4)$$
$$y(t) = \begin{cases} 1, & |t| < 1/4 \\ 0, & T_1 < |t| < 1/2 \end{cases}, \; y(t) = y(t+T)$$

$$x(t) \xleftrightarrow{FS} a_k, \quad a_0 = \frac{2T_1}{T}, \quad a_k = \frac{\sin(k\pi/2)}{k\pi} \text{ for } k \neq 0. \qquad (15.5)$$

$$y(t) \xleftrightarrow{FS} b_k, \quad b_k = \begin{cases} 0.5, & k = \pm 1 \\ 0, & \text{otherwise} \end{cases}. \qquad (15.6)$$

Given FS representations of $x(t)$ and $y(t)$, we can easily find the FS representation of our signal $z(t)$ by using linearity property of FS.

$$z(t) \xleftrightarrow{FS} c_k = a_k + jb_k. \qquad (15.7)$$

After finding FS representation or FS coefficients c_k's of our signal we time reversed and conjugated these coefficients and made a reconstruction. The result is shown in Figure 15.1. We observe that the reconstructed signal is the complex conjugate of our original signal.

Matlab code of this project is given below, in mfile "project15.m".

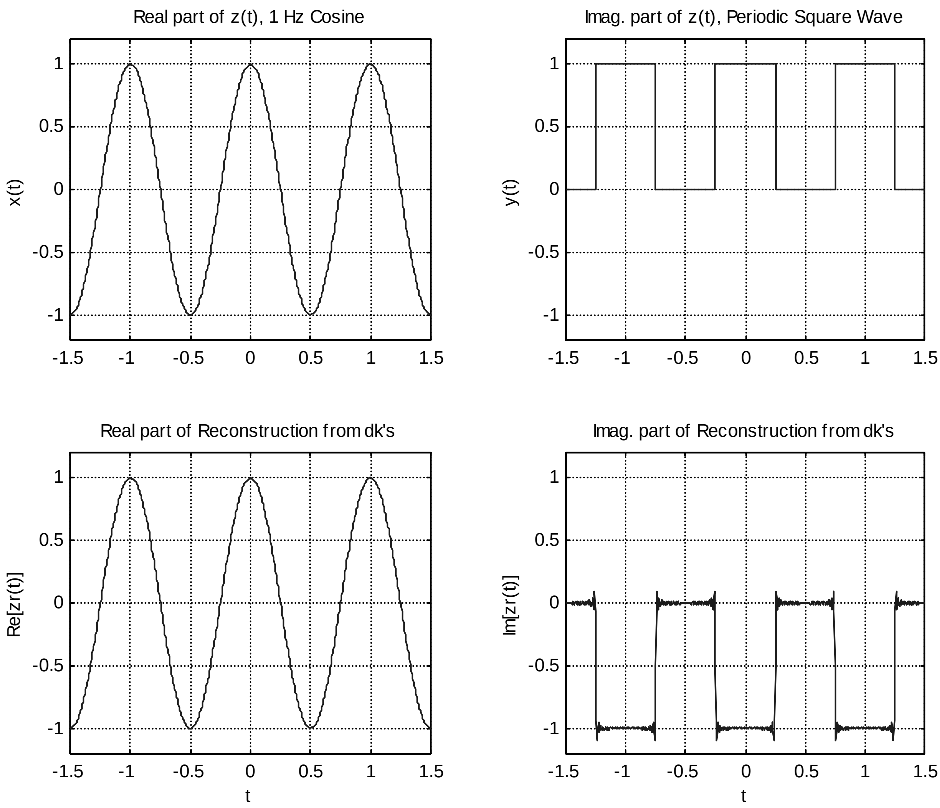

Figure 15.1 A complex periodic signal (upper panels), and reconstructed signal using time reversed and complex conjugated version of its FS coefficients (lower panels).

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% Project #15
% Title: Properties of Continuous Time Fourier Series
%        Conjugatioan and Conjugate Symmetry

% Generation of 1Hz cosine (Real part of our signal)
t=-1.5:0.005:1.5;
xt=cos(2*pi*t);

% Generation of periodic square wave (Imaginary part of
% our signal)
yt=xt>0;
```

```
% Our perodic complex valued signal
zt=xt+j*yt;

% FS coefficients of 1Hz cosine, xt (over k=-50..50)
ak=zeros(1,101);
ak(50)=0.5;ak(52)=0.5;

% FS coefficients of periodic square wave, yt
k=-50:50;
T=1;T1=0.25;
bk=sin(k*2*pi*(T1/T))./(k*pi);
% Manual correction for a0 -> ak(51)
bk(51)=2*T1/T;

% FS coefficients of zt (using linearty property of FS)
ck=ak+j*bk;

% Flip ck's and conjugate
dk=conj(fliplr(ck));

% Reconstruction from dk's with 101 terms (M=50)
w0=2*pi/T;zrt=zeros(1,length(t));
for k=-50:50
   zrt=zrt + dk(k+51)*exp(j*k*w0*t);
end

figure(1);set(gcf,'defaultaxesfontsize',8)
subplot(2,2,1);plot(t,xt);ylabel('x(t)')
title('Real part of z(t), 1 Hz Cosine')
set(gca,'ylim',[-1.2 1.2]);grid

subplot(2,2,2);plot(t,yt);ylabel('y(t)')
title('Imag. part of z(t), Periodic Square Wave')
set(gca,'ylim',[-1.2 1.2]);grid
```

```
subplot(2,2,3);plot(t,real(zrt))
xlabel('t');ylabel('Re[zr(t)]')
title('Real part of Reconstruction from dk''s')
set(gca,'ylim',[-1.2 1.2]);grid

subplot(2,2,4);plot(t,imag(zrt))
xlabel('t');ylabel('Im[zr(t)]')
title('Imag. part of Reconstruction from dk''s')
set(gca,'ylim',[-1.2 1.2]);grid
```