Project #6

Continuous Time Complex Exponential and Sinusoidal Signals

Exponential and sinusoidal signals play an important role in signal analysis, as we can express many other signals in terms of these signals. Here we will see how they look by generating and viewing some samples of them in Matlab.

Part 1: Real Exponential Signals

General continuous time complex exponential functions have the form

$$x(t) = Ce^{at}, \tag{6.1}$$

where C and a complex numbers in general. When both C and a are real, $x(t)$ becomes a real function.

Let us create some examples of real exponential signals over a time axis stretching from –1 to 1, for both positive and negative a values. We will take the value of C as 1, in all cases below.

```
» t=-1:0.01:1;                                          MCL 1
» x1=exp(t);x2=exp(2*t);                                MCL 2
» x3=exp(-t);x4=exp(-2*t);                              MCL 3
» plot(t,x1,t,x2,t,x3,t,x4);grid                        MCL 4
» xlabel('t');ylabel('x(t)')                            MCL 5
» title('Examples of Real Time Exponentials')           MCL 6
» legend('a=1','a=2','a=-1','a=-2')                     MCL 7
```

While plotting more than one x-y pair of variables or data, the plot utility automatically selects a different color for each pair. That is how we obtained the colorful picture shown in Figure 6.1. In figures with multiple plots, the *legend* utility comes to rescue to let us know which plot depicts what.

After doing a multiple plot, we just present the legend utility with the list of identification tags corresponding to the plots and it takes care of the rest. It creates a legend box that we can drag –with mouse– to wherever we like on the figure window.

Real exponentials with positive and negative *a* values are called *growing* and *decaying* exponentials respectively.

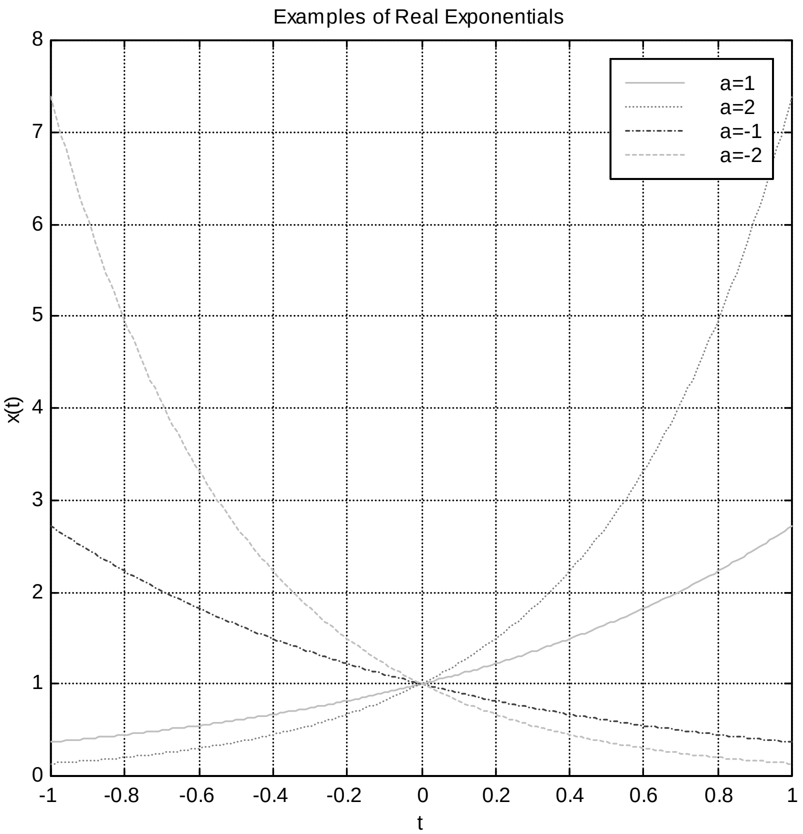

Figure 6.1 Some examples of real exponential signals.

We will show you some Matlab tricks now. The work that we have done by `MCL 1` through `MCL 4` could have been done by a single line. Amazing but true! Study and try to understand the following line where we utilized Matlab's capabilities to handle matrices heavily. Actually, this is more than what we have done, as it also includes the case of *a* being 0.

```
» t=-1:0.01:1;a=-2:2;x=exp(t'*a);plot(t,x);grid
```

Almost every argument to Matlab functions can be a single value, or a vector or a matrix, in which case the function tries to do the most reasonable act consistent with the specific task it was designed to do. Using vector or matrix arguments where possible is encouraged in Matlab, as this helps run Matlab codes much faster.

For instance, the plot function above has a vector of size *M* by 1 as its first argument (or the x-axis) and a matrix of size *N* by *M* as its second argument (or the y-axis), in which case it plotted each column of the second argument with respect to the first argument, one by one.

Can you now explain what the statement in the middle, i.e. `x=exp(t'*a);` does? (The apostrophe "'" sign denotes the transpose operation.)

Part 2: Periodic Complex Exponential and Sinusoidal Signals

When the parameter *a* in equation (6.1), i.e. in the expression of the general continuous time complex exponential function, is purely imaginary, then the exponential signal will be periodic. We can show this as follows.

Let $a = jw_0$, then $x(t)$ defined as Ce^{at} becomes

$$x(t) = Ce^{jw_0 t} , \tag{6.2}$$

and $x(t+T)$ becomes

$$x(t+T) = Ce^{jw_0(t+T)} = Ce^{jw_0 t}e^{jw_0 T} . \tag{6.3}$$

For the last expression to be equal to $x(t)$, $e^{jw_0 T}$ must be 1, which implies that

$$w_0 T = \pm 2\pi k , \tag{6.4}$$

where *k* is an integer. We therefore find the fundamental period of $x(t)$, T_0, as

$$T_0 = \pm \left.\frac{2\pi k}{w_0}\right|_{k=1} = \frac{2\pi}{|w_0|}. \tag{6.5}$$

We will now show that the complex signal $x(t)$ has sinusoidal real and imaginary parts. To that end, we express the complex number in polar form as

$$C = |C|e^{j\phi}. \tag{6.6}$$

Then by substituting equation (6.6) in equation (6.2) and using Euler's formula, $x(t)$ becomes

$$\begin{aligned} x(t) &= |C|e^{j\phi}e^{jw_0 t} = |C|e^{j(w_0 t+\phi)} \\ &= |C|\left[\cos(w_0 t+\phi) + j\sin(w_0 t+\phi)\right]. \end{aligned} \tag{6.7}$$

From equation (6.7) we figure out that

$$\begin{aligned} \mathrm{Re}\{x(t)\} &= |C|\cos(w_0 t+\phi) \\ \mathrm{Im}\{x(t)\} &= |C|\sin(w_0 t+\phi). \end{aligned} \tag{6.8}$$

This way, we have shown that the well-known general cosine or sine functions or signals are real and imaginary parts of a complex exponential signal with a purely imaginary exponent, respectively.

Let us now concentrate on the cosine signal of the form $x(t) = A\cos(w_0 t+\phi)$, where we have renamed the amplitude as A. We will change the parameters A (amplitude), w_0 (angular frequency) and ϕ (phase), and make several plots to see their effect on signal shape.

Below, we generate and plot four cosine signals with different set of parameters. Each time we changed one of three parameters of a reference cosine signal with $A = 1$, $w_0 = 2\pi$ and $\phi = 0$. The result is shown in Figure 6.2.

```
» t=-1:0.01:1;                                        MCL 8
» A=1;w0=2*pi;phi=0;x1=A*cos(w0*t+phi);               MCL 9
» A=2;w0=2*pi;phi=0;x2=A*cos(w0*t+phi);               MCL 10
» A=1;w0=4*pi;phi=0;x3=A*cos(w0*t+phi);               MCL 11
```

```
» A=1;w0=2*pi;phi=-pi/2;x4=A*cos(w0*t+phi);           MCL 12
» plot(t,[x1;x2;x3;x4]');grid                          MCL 13
» xlabel('t');ylabel('x(t)')                           MCL 14
» title('Examples of Cosine Signals')                  MCL 15
» legend('A=1,wo=2*pi,phi=0','A=2,wo=2*pi,phi=0','A=1,...
» wo=4*pi,phi=0','A=1,wo=2*pi,phi=-pi/2')              MCL 16
```

Note how we combined different cosine signals into a matrix at `MCL 13`. Note also the three dots "…" at `MCL 16`, this continuation feature enables us to divide long command lines into shorter ones.

As we observe in Figure 6.2, with the proper choice of the phase term ϕ, a cosine can become a sine and vice versa. Therefore both cosine and sine signals go under the name *sinusoidal signals*.

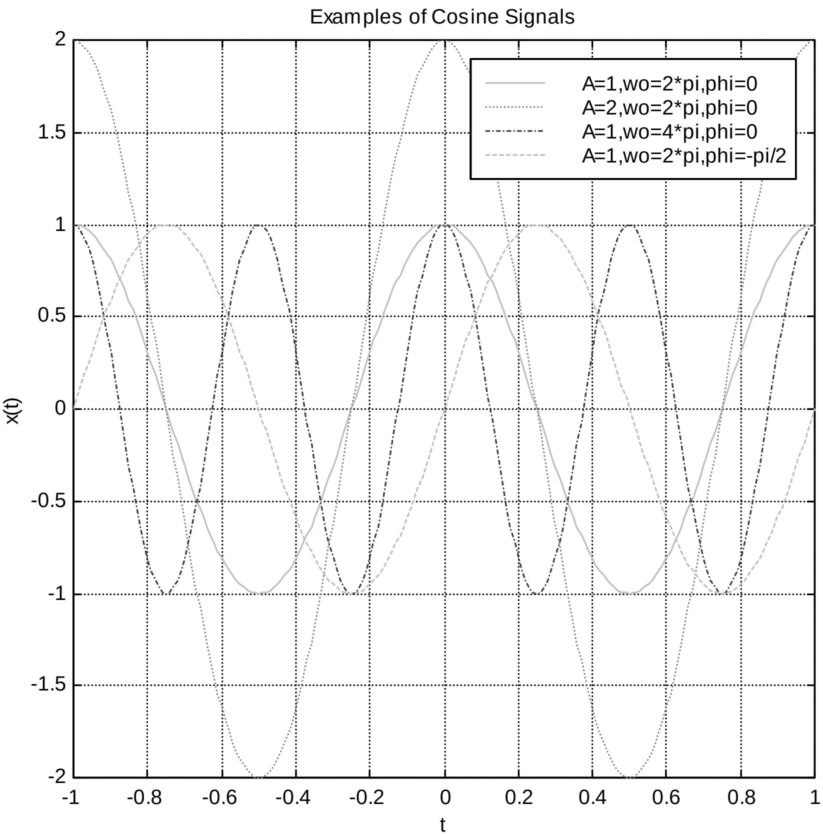

Figure 6.2 Some examples of cosine signals.

Part 3: Harmonically Related Complex Exponentials

Here, we will generate harmonically related periodic complex exponential signals defined as

$$\left\{\phi_{w_0,k}\right\}_{k=0,\pm1,\pm2,\ldots} = e^{jkw_0 t}. \tag{6.9}$$

A member of this set of functions (signals) for any particular value of k is periodic with a fundamental period $T_k = \dfrac{2\pi}{|k|\,w_0}$ and all the signals in this set are periodic with $T_0 = 2\pi / w_0$. We should note here that for any different value of angular frequency w_0, we have a different set of harmonically related signals. Notation used in equation (6.9) clearly emphasizes this point. Another commonly used notation to denote harmonically related exponentials is to write down a member of this set for a particular value of k as $\phi_k(t) = e^{jkw_0 t}$.

Let us take w_0 as 2π and generate a few samples from the corresponding set, for $k = -2$, 0, 1, and 3. As these signals are complex valued, we will plot their real and imaginary parts separately. The result is shown in Figure 6.3.

```
» t=-1:0.01:1;w0=2*pi;                                   MCL 17
» k=-2;Phi_2=exp(j*k*w0*t);                              MCL 18
» k=0;Phi0=exp(j*k*w0*t);                                MCL 19
» k=1;Phi1=exp(j*k*w0*t);                                MCL 20
» subplot(2,1,1)                                         MCL 21
» plot(t,real(Phi_2),'k',t,real(Phi0),'k:',...
t,real(Phi1),'k-.')                                      MCL 22
» set(gca,'ylim',[-1.1 1.1])                             MCL 23
» title('Harmonically Related Exponentials')             MCL 24
» ylabel('Real Part')                                    MCL 25
» legend('k=-2','k=0','k=1')                             MCL 26
» subplot(2,1,2)                                         MCL 27
» plot(t,imag(Phi_2),'k-',t,imag(Phi0),'k:',...
t,imag(Phi1),'k-.')                                      MCL 28
```

```
» set(gca,'ylim',[-1.1 1.1])                         MCL 29
» xlabel('t');ylabel('Imaginary Part')               MCL 30
```

Note how we set the y-axes limits at `MCL 23` and `MCL 29`, by using the *ylim* property of axes objects[1]. Note also the use of plot utility at `MCL 22` and `MCL 28`, the string arguments following an x-y pair indicates the color and type of line or marker to be used to plot that particular pair. For instance, the string `'k-'` indicates that the preceding pair will be plotted using a black solid marker. For other marker (color and type) codes refer to the plot utility's help.

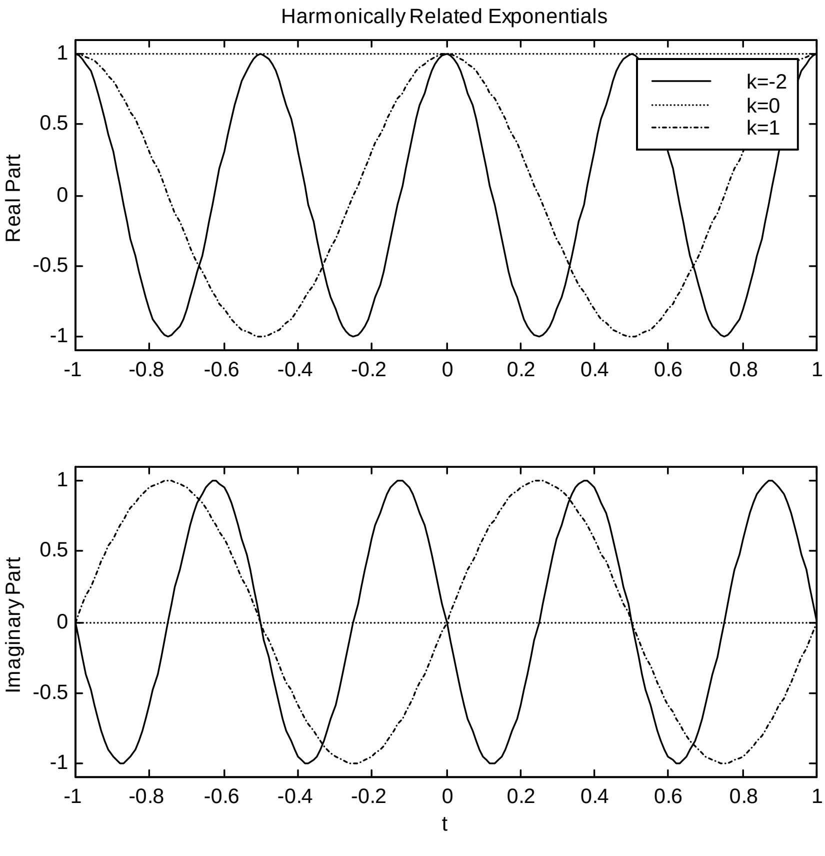

Figure 6.3 Real (upper panel) and imaginary (lower panel) parts of harmonically related exponentials for $w_0 = 2\pi$ and $k = -2$, 0, and 1.

[1] To see other properties of axes objects, type "get(gca)" at the Matlab prompt, where *gca* refers to the handle of current axes object.

Part 4: General Complex Exponential Signals

Here we will study the complex exponential signal $x(t) = Ce^{at}$ in its most general form, where both C and a complex numbers. We express C in polar form as in equation (6.6), $C = |C|e^{j\phi}$ and a in rectangular form as

$$a = r + jw_0 \tag{6.10}$$

Then by substituting equations (6.6) and (6.10) into (6.1), $x(t)$ becomes

$$\begin{aligned} x(t) &= |C|e^{j\phi}e^{(r+jw_0)t} = |C|e^{rt}e^{j(w_0 t+\phi)} \\ &= |C|e^{rt}\left[\cos(w_0 t+\phi) + j\sin(w_0 t+\phi)\right] \\ &= |C|e^{rt}\cos(w_0 t+\phi) + j|C|e^{rt}\sin(w_0 t+\phi) \end{aligned} \quad . \tag{6.11}$$

We observe that both real and imaginary parts of the complex exponential signal are sinusoidal bounded by an envelope $|C|e^{rt}$. Let us now generate a few samples of this signal with growing and decaying exponential envelopes. For the first sample we will select $C = e^{j\pi/4}$ and $r = 1 + j4\pi$, hence it will have a growing exponential envelope.

```
» t=-1:0.01:1;                                        MCL 31
» C=1*exp(j*pi/4);                                    MCL 32
» r=1;w0=4*pi;a=r+j*w0;                               MCL 33
» xt=C*exp(a*t);                                      MCL 34
» xenvp=abs(C)*exp(r*t);                              MCL 35
» xenvn=-abs(C)*exp(r*t);                             MCL 36
» plot(t,real(xt),'k',t,imag(xt),'k:',...
t,xenvp,'k--',t,xenvn,'k--')                          MCL 37
» title('Complex Exponential');xlabel('t')            MCL 38
» legend('Real part','Imag. part')                    MCL 39
```

The result is shown in Figure 6.4 (growing exponential envelope).

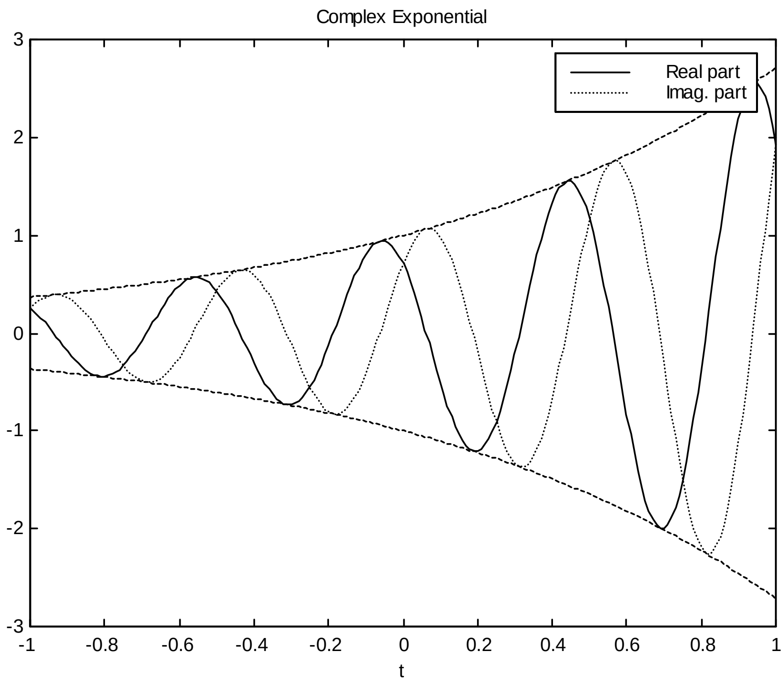

Figure 6.4 Real and imaginary parts of a complex exponentials with $C = \sqrt{2} + j\sqrt{2}$ and $r = 1 + j4\pi$.

The next sample will have a decaying exponential envelope with $C = e^{j\pi/4}$ and $r = -1 + j4\pi$.

```
» t=-1:0.01:1;                                      MCL 40
» C=1*exp(j*pi/4);                                  MCL 41
» r=-1;w0=4*pi;a=r+j*w0;                            MCL 42
» xt=C*exp(a*t);                                    MCL 43
» xenvp=abs(C)*exp(r*t);                            MCL 44
» xenvn=-abs(C)*exp(r*t);                           MCL 45
» plot(t,real(xt),'k',t,imag(xt),'k:',...
t,xenvp,'k--',t,xenvn,'k--')                        MCL 46
» title('Complex Exponential');xlabel('t')          MCL 47
» legend('Real part','Imag. part')                  MCL 48
```

The result is shown in Figure 6.5 (decaying exponential envelope).

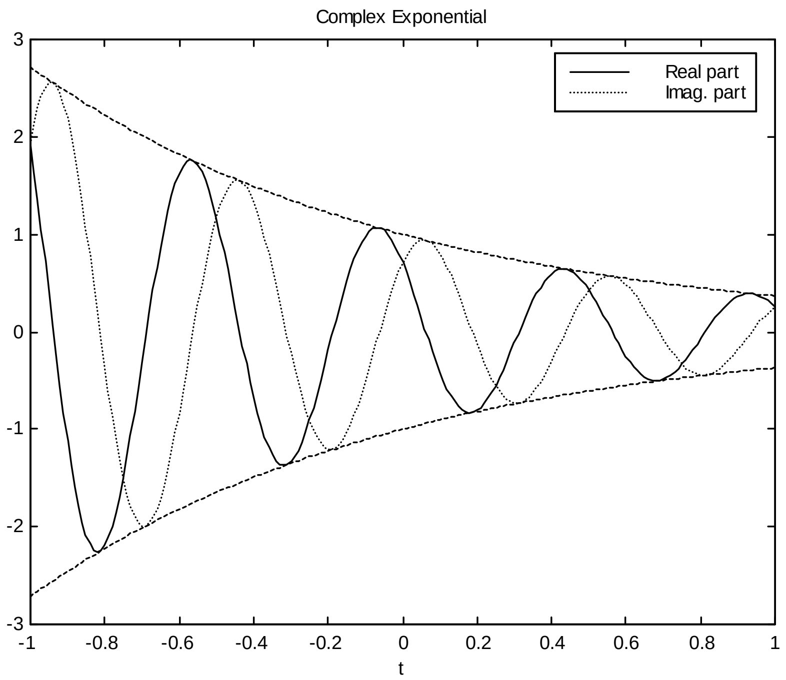

Figure 6.5 Real and imaginary parts of a complex exponentials with $C = \sqrt{2} + j\sqrt{2}$ and $r = -1 + j4\pi$.

Note that at `MCL 32` or `MCL 41`, instead of the polar form, we could have also used the rectangular form to express *C* as `C=sqrt(2)+j*sqrt(2)`.