Project #7

Discrete Time Complex Exponential and Sinusoidal Signals

We will now explore discrete time complex exponential and sinusoidal signals. Compared with the continuous time case that we dealt with in Project #6, things will be much easier here, except maybe Part 3, where we will discuss discrete time harmonically related signal.

Part 1: Real Exponential Signals

General discrete time complex exponential functions have the form

$$x[n] = Ce^{\beta n} \tag{7.1}$$

where C and a complex numbers in general. We can slightly modify this definition by calling e^{β} as α, to obtain

$$x[n] = Ce^{\beta n} = C(e^{\beta})^n = C\alpha^n . \tag{7.2}$$

The form of discrete time complex exponential functions expressed by equation (7.2) is more common. When both C and α are real, $x[n]$ becomes a real function. Let us create some examples of real exponential signals over indices from –5 to 5, for some different α values. We will take the value of C as 1, in all cases below.

```
» n=-8:8;                                MCL 1
» alp=1.2;xn1=alp.^n;                    MCL 2
» subplot(2,2,1);stem(n,xn1)             MCL 3
» set(gca,'xlim',[-8.5 8.5])             MCL 4
» xlabel('n');ylabel('x1[n]')            MCL 5
» title('C=1,alpha=1.2')                 MCL 6
» alp=0.8;xn2=alp.^n;                    MCL 7
» subplot(2,2,2);stem(n,xn2)             MCL 8
» set(gca,'xlim',[-8.5 8.5])             MCL 9
» xlabel('n');ylabel('x2[n]')            MCL 10
```

```
» title('C=1,alpha=0.8')                          MCL 11
» alp=-0.8;xn3=alp.^n;                            MCL 12
» subplot(2,2,3);stem(n,xn3)                      MCL 13
» set(gca,'xlim',[-8.5 8.5])                      MCL 14
» xlabel('n');ylabel('x3[n]')                     MCL 15
» title('C=1,alpha=-0.8')                         MCL 16
» alp=-1.2;xn4=alp.^n;                            MCL 17
» subplot(2,2,4);stem(n,xn4)                      MCL 18
» set(gca,'xlim',[-8.5 8.5])                      MCL 19
» xlabel('n');ylabel('x4[n]')                     MCL 20
» title('C=1,alpha=-1.2')                         MCL 21
```

The result is shown in Figure 7.1. What happens when α is 1 or –1?

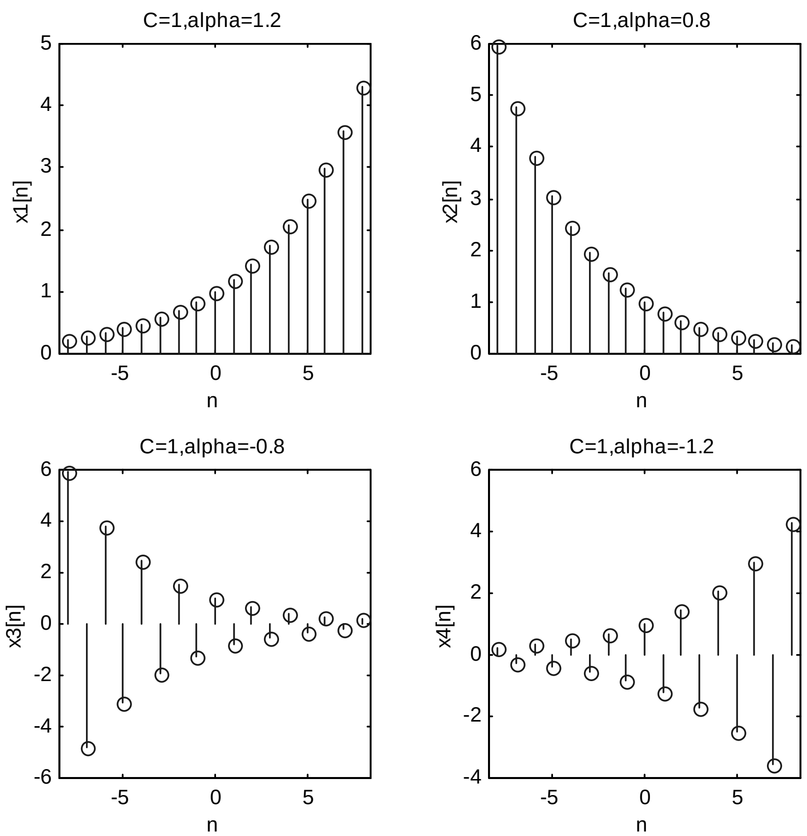

Figure 7.1 Some examples of real exponential signals.

Part 2: Sinusoidal Signals

We will now study discrete time complex exponential signal when the parameter β in equation (7.1) is purely imaginary. Let $\beta = jw_0$ then $x[n]$ defined as $Ce^{\beta n}$ becomes

$$x[n] = Ce^{jw_0 n}\ . \tag{7.2}$$

A signal of this form will have sinusoidal real and imaginary parts, but interestingly this does not mean that it will be periodic. As opposed to continuous time sinusoidal signals, discrete time sinusoidal signals do not need to be periodic. To study periodicity properties of a signal given by equation (7.2), let us evaluate $x[n+N]$, N being the supposed period, as

$$x[n+N] = Ce^{jw_0(n+N)} = Ce^{jw_0 n}e^{jw_0 N}\ . \tag{7.3}$$

We should note here N can only be an integer, as we are in discrete time where signals are defined only at integer index values.

For the last expression to be equal to $x[n]$, $e^{jw_0 N}$ must be 1, which implies that

$$w_0 N = \pm 2\pi k\ , \tag{7.4}$$

where k is an integer. From equation (7.4) we find that $w_0 / 2\pi$ must be a rational number, as N must be an integer.

$$\frac{w_0}{2\pi} = \frac{\pm k}{N}\ . \tag{7.5}$$

Therefore, if it exits, we can find the fundamental period of $x[n]$, N_0, as

$$N_0 = \left.\frac{2\pi k}{|w_0|}\right|_{\substack{\text{smallest } k \text{ that} \\ \text{makes } N_0 \text{ integer}}}\ . \tag{7.6}$$

We will now show that the complex signal $x[n]$ has sinusoidal real and imaginary parts. To that end, as we did in Project #6, we express the complex number in polar form as

$$C = |C|e^{j\phi}. \tag{7.7}$$

Then, by substituting equation (7.7) in equation (7.2) and using Euler's formula, we obtain

$$\begin{aligned} x[n] &= |C|e^{j\phi}e^{jw_0 n} = |C|e^{j(w_0 n+\phi)} \\ &= |C|\left[\cos(w_0 n+\phi) + j\sin(w_0 n+\phi)\right]. \end{aligned} \tag{7.8}$$

Equation (7.8) indicates that

$$\begin{aligned} \mathrm{Re}\{x[n]\} &= |C|\cos(w_0 n+\phi) \\ \mathrm{Im}\{x[n]\} &= |C|\sin(w_0 n+\phi). \end{aligned} \tag{7.9}$$

Let us now concentrate on the cosine signal of the form $x[n] = A\cos(w_0 n+\phi)$, where we have renamed the amplitude as A. We will change the parameter w_0 to see its effect on periodicity. We will take A as 1, ϕ as 0 and w_0 as 1, $\pi/4$ and $3\pi/4$ and generate and plot the corresponding signals.

For the first one, i.e. the cosine with w_0 = 1, we will also plot an envelope to better see that the signal is not periodic.

```
» n=-10:10;                                            MCL 22
» w0=1;xn1=cos(w0*n);                                  MCL 23
» stem(n,xn1);hold on                                  MCL 24
» t=-10:0.1:10;                                        MCL 25
» xt=cos(t);                                           MCL 26
» plot(t,xt,'r');hold off                              MCL 27
» axis([-10.5 10.5 -1 1]);xlabel('n');ylabel('x[n]')   MCL 28
» title('A Nonperiodic Discrete Time Cosine Signal')   MCL 29
```

The *hold*[1] command was used to keep the figure contents intact, enabling us to overlay plots. As the envelope, we used a continuous time cosine –with the same angular frequency, amplitude, and phase– over a time interval from –10 to 10, with a finer axis to simulate continuous time.

This first cosine along with its envelope is shown in Figure 7.2.

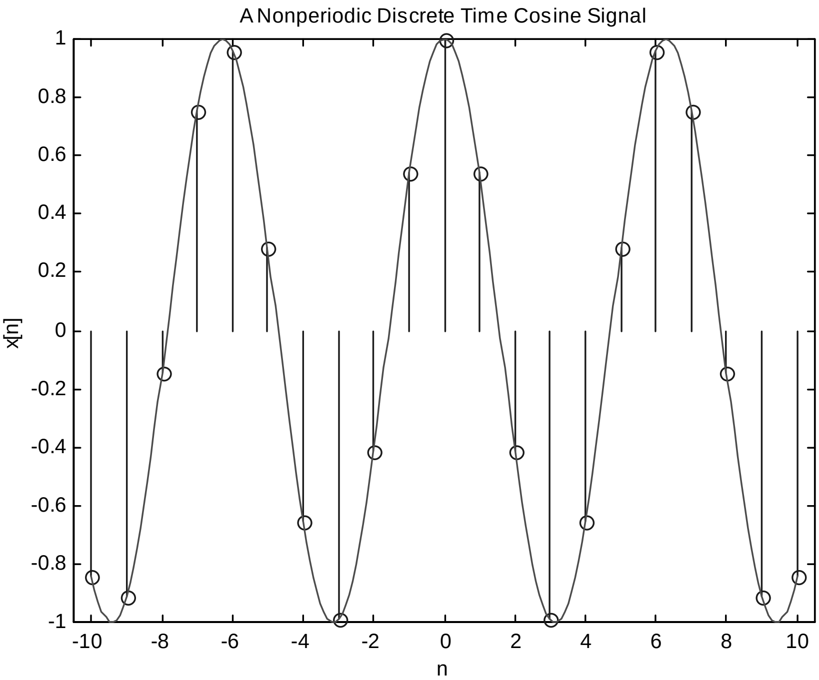

Figure 7.2 A non-periodic discrete time cosine signal cos(n), i.e. $A\cos(w_0 n + \phi)$, with $A = 0$, $w_0 = 1$, and $\phi = 0$. The envelope signal is cos(t).

As the index n already in Matlab's memory or workspace, we will not regenerate it to make the plots of the second and third cosines with $w_0 = \pi/4$ and $w_0 = 3\pi/4$ respectively. These two cosines are shown in Figure 7.3.

```
» w0=pi/4;xn2=cos(w0*n);                              MCL 30
» subplot(2,1,1);stem(n,xn2)                          MCL 31
```

[1] "hold on" holds the current plot and all axis properties so that subsequent graphing commands add to the existing graph. "Hold off" returns to the default mode whereby "plot" commands erase the previous plots and reset all axis properties before drawing new plots. "hold", by itself, toggles the hold state.

```
» title('Discrete Time Cosine Signals')              MCL 32
» w0=3*pi/4;xn3=cos(w0*n);                           MCL 33
» subplot(2,1,2);stem(n,xn3);xlabel('n')             MCL 34
```

Although they have different angular frequencies, both cosine signals in Figure 7.3 are periodic with period 8. How do you explain this? Redo these cosine plots with their envelopes, as we did with the first cosine plot, and use the formula given by equation (7.6) to see what is going on.

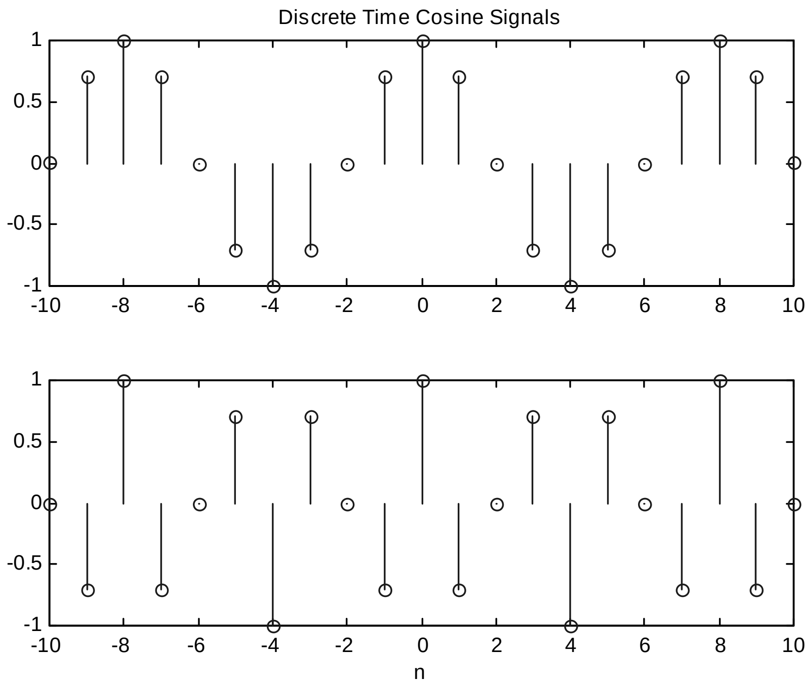

Figure 7.3 Two periodic discrete time cosine signals, $\cos(\frac{\pi}{4}n)$ (upper panel) and $\cos(\frac{3\pi}{4}n)$ (lower panel). Both signals have a period of 8.

Part 3: Harmonically Related Complex Exponentials

Here, we will generate harmonically related periodic complex exponential signals defined as

$$\left\{\phi_{N,k}\right\}_{k=0,\pm1,\pm2,\dots} = e^{jk(2\pi/N)n}. \tag{7.10}$$

For any different value of period *N*, we have a different set of harmonically related signals. The notation used in equation (7.10) clearly emphasizes this point. Another commonly used notation to denote harmonically related exponentials is to write down a member of this set for a particular value of *k* as $\phi_k[n] = e^{jk(2\pi/N)n}$.

All the member of this set of signals are periodic with a period *N*. We can verify this by either using equation (7.6) or directly by showing that $\phi_k[n]$ and $\phi_k[n+N]$ are equal.

$$\begin{aligned}\phi_k[n+N] &= e^{jk(2\pi/N)(n+N)} = e^{jk(2\pi/N)n} e^{jk(2\pi/N)N} \\ &= e^{jk(2\pi/N)n} \underbrace{e^{j2\pi k}}_{1} = e^{jk(2\pi/N)n} = \phi_k[n]\end{aligned} \quad (7.11)$$

As opposed the continuous time harmonically related exponentials which have infinitely many members, for any choice of w_0, discrete time harmonically related exponentials do have only *N* distinct members (signals) for any choice of *N*. We can prove this by showing that $\phi_{k+N}[n]$ and $\phi_k[n]$ are equal.

$$\begin{aligned}\phi_{k+N}[n] &= e^{j(k+N)(2\pi/N)n} = e^{jk(2\pi/N)n} e^{jN(2\pi/N)n} \\ &= e^{jk(2\pi/N)n} \underbrace{e^{j2\pi n}}_{1}\end{aligned} \quad . \quad (7.12)$$

Let us take *N* as 6 and generate the corresponding 6 harmonically related exponentials, i.e. $\phi_k[n]$ functions for *k* = 0, 1, 2 ... 5. We will plot real parts of these complex valued signals only.

```
» n=-12:12;N=6;w0=2*pi/N;                              MCL 35
» k=0;Phi0n=exp(j*w0*k*n);                             MCL 36
» subplot(3,2,1);stem(n,real(Phi0n));xlabel('n')       MCL 37
» axis([-12.5 12.5 -1.1 1.1]);ylabel('Phi0')           MCL 38
» k=1;Phi1n=exp(j*w0*k*n);                             MCL 39
» subplot(3,2,2);stem(n,real(Phi1n));xlabel('n')       MCL 40
» axis([-12.5 12.5 -1.1 1.1]);ylabel('Phi1')           MCL 41
```

```
» k=2;Phi2n=exp(j*w0*k*n);                              MCL 42
» subplot(3,2,3);stem(n,real(Phi2n));xlabel('n')         MCL 43
» axis([-12.5 12.5 -1.1 1.1]);ylabel('Phi2')            MCL 44
» k=3;Phi3n=exp(j*w0*k*n);                              MCL 45
» subplot(3,2,4);stem(n,real(Phi3n));xlabel('n')         MCL 46
» axis([-12.5 12.5 -1.1 1.1]);ylabel('Phi3')            MCL 47
» k=4;Phi4n=exp(j*w0*k*n);                              MCL 48
» subplot(3,2,5);stem(n,real(Phi4n));xlabel('n')         MCL 49
» axis([-12.5 12.5 -1.1 1.1]);ylabel('Phi4')            MCL 50
» k=5;Phi5n=exp(j*w0*k*n);                              MCL 51
» subplot(3,2,6);stem(n,real(Phi5n));xlabel('n')         MCL 52
» axis([-12.5 12.5 -1.1 1.1]);ylabel('Phi5')            MCL 53
```

The result is shown in Figure 7.4.

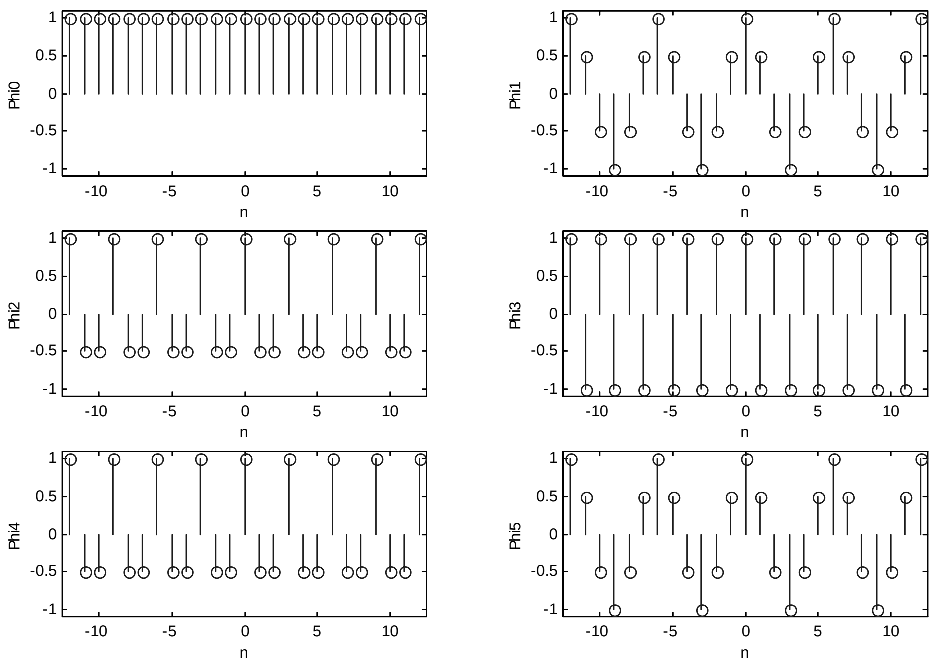

Figure 7.4 Discrete time harmonically related exponentials, $\phi_k[n] = e^{jk(2\pi/N)n}$, for $N = 6$ and $k = 0,1,2, .. 5$.

Part 4: General Complex Exponential Signals

Here we will study the discrete time complex exponential signal $x[n]=C\alpha^n$ in its most general form, where both C and α complex numbers. We will express both C and α in polar form. For C, we did this already in equation (7.7), $C=|C|e^{j\phi}$ and for α,

$$\alpha=|\alpha|e^{jw_0}\,. \tag{7.13}$$

Then by substituting equations (7.7) and (7.13) into (7.2), $x[n]$ becomes

$$\begin{aligned} x[n]=C\alpha^n &=|C|e^{j\phi}\left(|\alpha|e^{jw_0}\right)^n=|C|e^{j\phi}\,|\alpha|^n\,e^{jw_0 n} \\ &=|C||\alpha|^n\,e^{j(w_0 n+\phi)} \\ &=|C||\alpha|^n\cos(w_0 n+\phi)+j\,|C||\alpha|^n\sin(w_0 n+\phi) \end{aligned}\,. \tag{7.14}$$

We observe that both real and imaginary parts of the complex exponential signal are sinusoidal bounded by an envelope $|C||\alpha|^n$. Let us now generate a few samples of this signal with growing and decaying exponential envelopes. For the first sample we will select $C=e^{j\pi/4}$ and $\alpha=1.1e^{j\pi/4}$, hence it will have a growing exponential envelope.

```
» n=-12:12;                                          MCL 54
» C=1*exp(j*pi/2);                                   MCL 55
» w0=pi/4;alp=1.1*exp(j*w0);                         MCL 56
» xn=C*alp.^n;                                       MCL 57
» stem(n,real(xn),'k');hold on                       MCL 58
» set(gca,'xlim',[-12.5 12.5])                       MCL 59
» t=-12:0.1:12;                                      MCL 60
» xenvp=abs(C)*abs(alp).^t;                          MCL 61
» xenvn=-xenvp;                                      MCL 62
» plot(t,xenvp,'k--',t,xenvn,'k--');hold off         MCL 63
» title('Real Part of Complex Exponential')          MCL 64
» xlabel('n');ylabel('x[n]')                         MCL 65
```

The real part of this exponential, which is a sinusoidal with a growing exponential envelope, is shown in Figure 7.5.

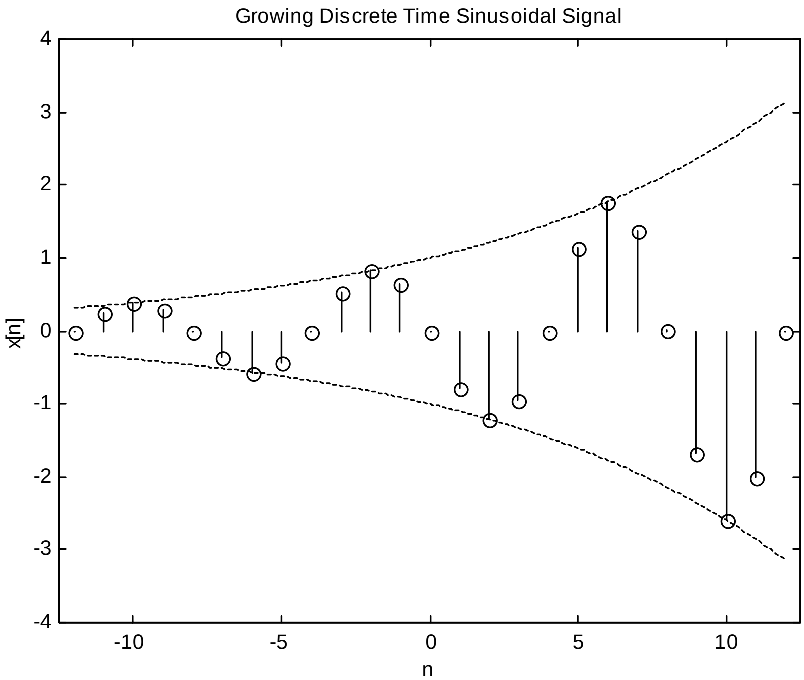

Figure 7.5 Real part of a complex exponential with $C = e^{j\pi/4}$ and $\alpha = 1.1e^{j\pi/4}$.

The next sample will have a decaying exponential envelope with $C = e^{j\pi/4}$ and $\alpha = 0.9e^{j\pi/4}$.

```
» alp=0.9*exp(j*w0);                                MCL 66
» xn=C*alp.^n;                                      MCL 67
» stem(n,real(xn),'k');hold on                      MCL 68
» set(gca,'xlim',[-12.5 12.5])                      MCL 69
» t=-12:0.1:12;                                     MCL 70
» xenvp=abs(C)*abs(alp).^t;                         MCL 71
» xenvn=-xenvp;                                     MCL 72
» plot(t,xenvp,'k--',t,xenvn,'k--');hold off        MCL 73
» title('Real Part of Complex Exponential')         MCL 74
» xlabel('n');ylabel('x[n]')                        MCL 75
```

The real part of this exponential, which is a sinusoid with a decaying exponential envelope, is shown in Figure 7.6.

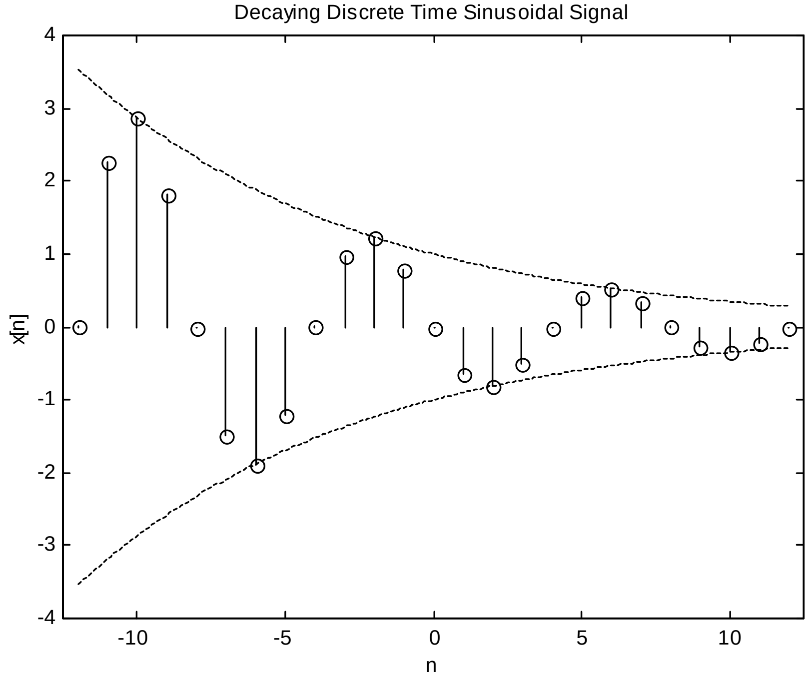

Figure 7.6 Real part of a complex exponential with $C = e^{j\pi/4}$ and $\alpha = 0.9e^{j\pi/4}$.