Project #8

Fourier Series Representation of Continuous Time Periodic Signals

A signal expressed by the formula

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t} \tag{8.1}$$

is periodic with period *T*, as it is a linear combination of *harmonically related complex exponentials* that are all periodic with *T*. To see that the family of functions

$$\phi_k(t) = e^{jkw_0 t} = e^{jk(2\pi/T)t},\ k = 0, \pm 1, \pm 2, \ldots \tag{8.2}$$

are all periodic with *T*, it suffices to show that $\phi_k(t) = \phi_k(t+T)$ for any integer *k*. Let us now do this by working out $\phi_k(t+T)$,

$$\phi_k(t+T) = e^{jk(2\pi/T)(t+T)} = e^{jk(2\pi/T)t} e^{jk2\pi} = e^{jk(2\pi/T)t} = \phi_k(t). \tag{8.3}$$

In equation (8.3), we used the fact that $e^{jk2\pi} = 1$ is for any integer *k*.

A signal that is a linear combination of harmonically related complex exponentials is periodic and the reverse is true too. That is, any *well-behaving periodic function*[1] can be expressed a linear combination of harmonically related complex exponentials, as in equation (8.1). The representation of a periodic signal in this way is referred to as *Fourier series representation*. The weights, a_k's, in this representation are referred as *Fourier series coefficients*. Given a periodic function *x*(*t*), it is possible determine its Fourier series coefficients through the following integral.

$$a_k = \int_T x(t) e^{-jkw_0 t} dt = \int_T x(t) e^{-jk(2\pi/T)t} dt \tag{8.4}$$

[1] By "well-behaving periodic function" we mean a periodic function that satisfies Dirichlet conditions. Please refer to section 3.4 "Convergence on Fourier Series" in your book, for more information.

This integral can be done over any time interval of length *T*, the period of the signal $x(t)$.

To demonstrate the first concept, i.e. a linear combination of harmonically related complex exponentials leads to a periodic function, we will work out the following example in Matlab.

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t} \tag{8.5}$$

where a_0=1, $a_1 = a_{-1} = \frac{1}{4}$, $a_2 = a_{-2} = \frac{1}{2}$, and $a_3 = a_{-3} = \frac{1}{3}$. Can you tell what the period of this function is?

This time we will put our Matlab command lines into an *mfile* and name it as "project8.m". You can use any text editor or Matlab's own editor to create an mfile. When you call this file within Matlab, all the lines in this file will be interpreted (executed) one by one automatically, in the order you put them in the file. For Matlab to find and execute this file you have to either switch Matlab's working directory to the directory where you saved your file into, using the *cd* (change dir) command or add the path of the file to the *Matlab path*. Below is the listing of that project8.m file. The result of this code is shown in Figure 8.1.

```
%Project #8
%Title: Fourier Series Representation of Continuous Time Periodic
%       Signals

% (Simulated) continuous time axis
t=-3:0.01:3;

% Component for k=0 (average value)
x0=1;
% First harmonic components for k=-1 and k=1
x1=(1/4)*exp(j*(-1)*2*pi*t)+(1/4)*exp(j*(1)*2*pi*t);
```

```
% Sum of mean and first components
y1=x0+x1;

% Second harmonic components for k=-2 and k=2
x2=(1/2)*exp(j*(-2)*2*pi*t)+(1/2)*exp(j*(2)*2*pi*t);

% Sum of all components up to 2nd order
y2=y1+x2;

% Third harmonic components for k=-3 and k=3
x3=(1/3)*exp(j*(-3)*2*pi*t)+(1/3)*exp(j*(3)*2*pi*t);

% Our signal, which is the sum of all components
x=x0+x1+x2+x3;

% We now can use subplot commands to see harmonic components
% and reconstruction stages of our signal

% Choose the first part to plot x1(t)
subplot(3,2,1);plot(t,x1)
axis([-3 3 -2 2]);title('x1(t)')

% Choose the second part to plot y1(t)=x0(t)+x1(t)
subplot(3,2,2);plot(t,y1)
axis([-3 3 -0.2 2]);title('x0(t)+x1(t)')

% We go on to plot x2(t) and x3(t) and x(t)
subplot(3,2,3);plot(t,x2)
axis([-3 3 -2 2]);title('x2(t)')
subplot(3,2,4);plot(t,y2)
axis([-3 3 -1 3]);title('x0(t)+x1(t)+x2(t)')
subplot(3,2,5);plot(t,x3);xlabel('t')
axis([-3 3 -1 1]);title('x3(t)')
subplot(3,2,6);plot(t,x);xlabel('t')
axis([-3 3 -1 4]);title('x(t)=x0(t)+x1(t)+x2(t)+x3(t)')
```

Note the use of comments starting with the percent sign % and also try to locate and study features and language constructs of Matlab that are new to you. You can modify this file, for example try changing the values of coefficients a_k's. In the example above, a_k's were chosen to be symmetric about the index $k = 0$, i.e. $a_k = a_{-k}$. Now, select some new a_k's to alter this symmetry and form the new signal. What do you observe? Is $x(t)$ a real signal when the coefficients are not symmetric?

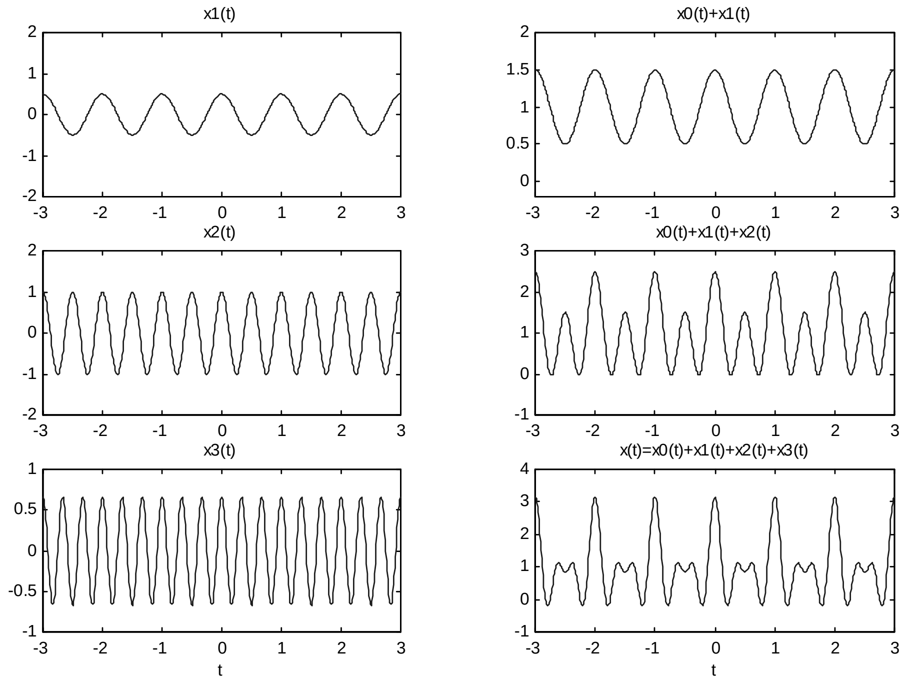

Figure 8.1 Demonstration of Fourier series representation of signals.

Now we will present you with another version of "project8.m" mfile where we utilized a *for loop*. Programming-wise the approach introduced in this version is more efficient. In this case, however, we will not be able to see individual construction steps of our signal.

Carefully review this new code, can you tell what the 2nd line from the end that assigns `x` to the real part of `x`, is doing? How about the indexing trick we used in the for loop, why did we put down `ak(k+4)` instead of `ak(k)`?

```
%Project #8 (Version 2)
%Title: Fourier Series Representation of Continuous Time Periodic
%       Signals

% index k
k=-3:3;

% coefficients
ak=[1/3 1/2 1/4 1 1/4 1/2 1/3];

% (Simulated) continuous time axis
t=-3:0.01:3;

% Initialization of the vector to hold our signal
x = zeros(1,length(t));

% Construction of the signal from using Fourier series
for k=-3:3
   x=x+ak(k+4)*exp(j*k*2*pi*t);
end

x=real(x);
plot(t,x)
```